A Unified Dynamic Model and Control Synthesis for Robotic Manipulators with Geometric End-Effector Constraints

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A compact dynamic model and a hybrid position/force controller for a constrained robot manipulator, subject to a set of holonomic (integrable) constraints have been developed in this study. The joint-space dynamics (DAEs) has been transformed into the constraint-space model in which the system dynamics can be readily decomposed into two orthogonal subsystems ; the motion-controlled subsystem is specified in the direction tangential to the known constraint surfaces, and the force-controlled subsystem is regulated in the orthogonal direction. Also utilizing the transformed dynamics, we have presented a hybrid adaptive control law to simultaneously manipulate the end-effector position and the contact force. Further, by a Lyapunov theory, it has been shown that the corresponding closed-loop system is globally stable under the parametric uncertainties.

Key Words : Hybrid Control, Constrained Robot, Holonomic Constraints, Differentiable-algebraic Equations (DAEs), Force-controlled (Position-controlled) Subsystem, Contact Force, Asymptotic Stability, Closed Kinematic Chain, Tangential (Orthogonal) Subspace, Lyapunov Theory

Nomenclature

Symbols written in bold type denote vectors or matrices, while scalars are written normally :

$\Re^+$: A set of non-negative real number ; $\Re^+ := [0, +\infty)$

$\Re^n$: The n-dimensional vector space with real elements $\Re$

$\Re^{n\times m}$: A set of all real-valued $(n\times m)$ matrices

sup : The supremum, the least upper bound

$\|\boldsymbol{x}\|$: The Euclidean norm of a vector $\boldsymbol{x}$; $\|\boldsymbol{x}\| = [\boldsymbol{x}^T\boldsymbol{x}]^{1/2}$, $\forall \boldsymbol{x}\in\Re^n$

$\boldsymbol{A}>0(<0)$: A positive(negative) definite matrix $\boldsymbol{A}$

$\lambda_{\max}(\boldsymbol{A})$: The maximum eigenvalue of matrix $\boldsymbol{A}$; $\lambda_{\max}(\boldsymbol{A}) = \max_i\{\lambda_i(\boldsymbol{A})\}$, where $\lambda_i(\boldsymbol{A})$ is the i th eigenvalue of matrix $\boldsymbol{A}$

$\lambda_{\min}(\boldsymbol{A})$: The minimum eigenvalue of matrix $\boldsymbol{A}$; $\lambda_{\min}(\boldsymbol{A}) = \min_i\{\lambda_i(\boldsymbol{A})\}$

$\|\boldsymbol{A}\|$: The induced norm of a real matrix $\boldsymbol{A}\in\Re^{n\times m}$; $\|\boldsymbol{A}\| = [\lambda_{\max}(\boldsymbol{A}^T\boldsymbol{A})]^{1/2}$

C^p : A set of p-times continuously differentiable functions

$\boldsymbol{E}_{n\times n}$: An $(n\times n)$ identity matrix

$\boldsymbol{0}_n$: A n-dimensional null vector

$\boldsymbol{0}_{n\times n}$: A $(n\times n)$ null matrices

L_p : The function norm in the Lebesgue space ; Let $\boldsymbol{f}(t) : \Re^+ \rightarrow \Re^n$ be Lebesgue measurable function, then the L_p-norm $\|\boldsymbol{f}\|_p$ is defined as $\|\boldsymbol{f}\|_p = [\int_0^\infty \|\boldsymbol{f}(t)\|^p dt]^{1/p} < \infty$, for $p\in[1, \infty)$. When $p=\infty$, $\boldsymbol{f}\in L_\infty$ if and only if $\|\boldsymbol{f}\|_\infty = \sup_{t\in[0,+\infty)} \|\boldsymbol{f}(t)\| < \infty$

$(\circ)^c$: A complement of $(\circ)$

$rs(\boldsymbol{A})$: The range space of matrix $\boldsymbol{A}$ (or column space of $\boldsymbol{A}$)

$rk(\boldsymbol{A})$: The rank of matrix $\boldsymbol{A}$

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$ns(A)$: The null space (or kernel) of matrix A

1. Introduction

During the last decade, there has been considerable research on the motion control of unconstrained robot manipulators whose end-effector do not significantly contact with the external environment.(Ortega, et. al., 1989 ; Reed, et. al., 1989 ; Sadegh, et. al., 1990) In general, these methods are mainly concerned with a pure position control. However, practical applications of such a system to higher level tasks are severely limited due to its performance. For example, in the execution of the advanced tasks involved in flexible manufacturing systems, such as machining tasks (grinding, deburring, polishing, etc.), assembly operations, and various material handling, the motion of the robot end-effector is kinematically constrained in some directions. A constrained robotic system, which forms closed kinematic chains, provides higher flexibility and dexterity in performing complex tasks. In these cases, a position control alone could lead to excessive contact forces or loss of contact with external environments. Therefore, to greatly enhance system performance, it is necessary to control both the position of the end-effector and the contact forces exerted by the end-effector on the environment.

For facilitating the dynamic analysis and the control synthesis, the proper mathematical model of such a system should be first formulated. The dynamics of the constrained robot system was discussed in references.(Mason, 1981 ; Raibert, et. al., 1981 ; Khatib, 1987) Unfortunately, none has employed a unified framework. Among the various approaches to the hybrid control architecture, the impedance control and the position/force control have been extensively suggested in the literature. In the hybrid position/force control, the position/force can be directly controlled to achieve accurate tracking performance. In the impedance control approach,(Hogan, 1985 ; Kazerooni, et. al., 1986 ; Wen, et. al., 1991) the external environment is modeled as a mechanical impedance to produce compliant motion, and thus the contact force is manipulated indirectly as a result of the position control. Mason(Mason, 1981) identified the task constraints represented by natural and artificial constraints. Based on the decomposition of the task space, Raibert and Craig(Raibert, et. al., 1981) suggested a hybrid controller. And without parametric uncertainties in the robotic model, a class of hybrid control strategies have been proposed, for example, studies by Yoshikawa(Yoshikawa, 1987) and Mcclamroch and Wang.(McClamroch, et. al., 1988) From a practical point of view, any dynamical system contains various system uncertainties. This problem motivates the adaptive control approach. Although several hybrid adaptive control schemes for robot manipulators have been suggested to date,(Han, et. al., 1990 ; Carelli, et. al., 1990 ; Fossen, et. al., 1991) more studies need to be conducted to ascertain the global stability of the control system, which simultaneously guarantees the asymptotic stability of the end-effector positions and the contact forces.

The primary aim of this research is to provide a compact approach to dynamics formulation and control synthesis for the robotic manipulator with closed kinematic chain. The first task is to transforms the joint-space dynamics (DAEs) into the constraint-space model. The constraint frame is set up as a direct sum of the position-controlled subspace and the force-controlled subspace ; the position subspace spanned by tangential vectors and the force subspace generated by normal vectors. Next, based on the new dynamic model, we present a hybrid impedance controller guaranteeing the global stability of the closed-loop system. The generalized positions and the contact forces of the gripper are simultaneously regulated in two orthogonal directions ; the position control in the free directions and the force control in the constrained directions. Therefore, it is shown that the robot system can be globally stabilized by the proposed control algorithm.

The rest of this paper is organized as follows. First, the system dynamics and problem formulation will be addressed in Sec. 2. After that, a hybrid adaptive control law is introduced in Sec. 3. Finally, the conclusions are given in Sec. 4.

2. System Dynamics and Problem Formulation

2.1 Preliminaries and the joint-space dynamics

This section provides a unified formulation for the kinematics and the dynamics of physically constrained robot manipulators. Consider the robotic system interacting with external objects, as shown in Fig. 1. In order to describe the kinematic and dynamic relationships among the components of the closed chain mechanism, a set of coordinate systems are defined as follows:

$\Pi_r\{{}^r o - {}^r x\,{}^r y\,{}^r z\}$ is the reference (or absolute) coordinate system; $\Pi_e\{{}^e o - {}^e x\,{}^e y\,{}^e z\}$ is the end-effector frame; $\Pi_c\{{}^c o - {}^c x\,{}^c y\,{}^c z\}$ is the constraint coordinate system.

In what follows, the concept of a configuration manifold will be briefly discussed. First, consider an unconstrained motion of a robotic mechanism whose joint position vector is denoted by $\boldsymbol{q} \in \Re^n$. Let $\boldsymbol{p} = [\boldsymbol{r}^T \boldsymbol{\Psi}^T]^T \in \Re^{n_0} (n_0 \leq n)$ be a generalized position vector of the end-effector with respect to Π_r, which is consists of the Cartesian position vector $\boldsymbol{r}$ and the vector $\Psi = [\alpha\beta\gamma]^T$ associated with the three Euler parameters. For representing arbitrary positions and orientations, $\boldsymbol{p}$ is typically chosen as a 6-dimensional manifold. Throughout the study, we consider the nonredundant robotic manipulator, i.e., $n_0 = n$(taking $n_0 = 6$). As shown pictorially in Fig. 2, the rotational motion can be described by the three Euler angles. More specifically, the Euler angles are specified in terms of the images of the three parameters(α, β, γ) obtained by performing three elementary rotations of the body-attached frame Π_e with respect to the fixed frame Π_r in a right-handed sense, that is, rotating α angle about the z axis, then β angle about the new y axis, and finally γ angle about the new x axis. Then the resulting overall transformation is given in a 3×3 matrix as

$$\boldsymbol{R} = \begin{bmatrix} C_\alpha C_\beta & C_\alpha S_\beta S_\gamma - S_\alpha C_\gamma & C_\alpha S_\beta C_\gamma + S_\alpha S_\gamma \\ S_\alpha C_\beta & S_\alpha S_\beta S_\gamma + C_\alpha C_\gamma & S_\alpha S_\beta C_\gamma - C_\alpha S_\gamma \\ -S_\beta & C_\beta S_\gamma & C_\beta C_\gamma \end{bmatrix} \quad (1)$$

where for convenience, $c_\alpha = \cos(\alpha)$, $s_\beta = \sin(\beta)$, and $c_\gamma = \cos(\gamma)$, and so on. Thus the orthogonal rotation matrix $\boldsymbol{R}$ maps the vectors from Π_e into Π_r. For the present study, the Euclidean motion of a rigid body (or frame) in a three-dimensional workspace can be specified by(you, 1994)

$$(\boldsymbol{r}, \boldsymbol{R}) \in \Re^3 \times SO(3) = SE(3)$$

where the Special Orthogonal group of order 3, which is denoted by the Lie group $SO(3) (\subset \Re^{3\times 3})$, represents a set (or a group) of all proper 3×3

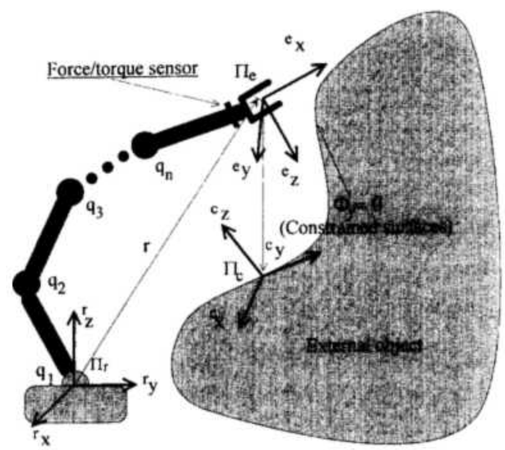

Fig. 1 Schematic diagram of a constrained robot system

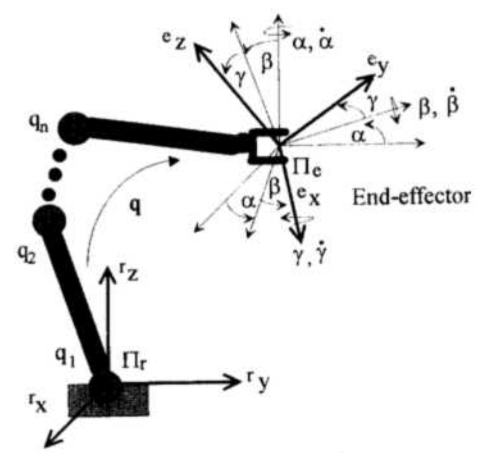

Fig. 2 Geometrical representation of Euler angles (yaw, pitch, and roll)

rotational matrices on $\Re^3$ and is a three-dimensional submanifold of $\Re^9$. And $SO(3)$ can be formally defined as

$$SO(3)=\{\boldsymbol{R}\in\Re^{3\times3}: det(\boldsymbol{R})=+1,\ \boldsymbol{R}^T\boldsymbol{R}=\boldsymbol{R}\boldsymbol{R}^T=E_{3\times3}\}$$

Consequently, the Special Euclidean group denoted by $SE(3)$ with dimension 6 can be considered as the configuration space for rigid bodies.

Now, $\bar{\omega}\in\Re^3$ denotes the angular velocity vector of the frame Π_e with respect to Π_r, and the time derivative of the orientation vector is called the Euler rates ($\dot{\Psi}$). Then the kinematic relationship between the angular velocity and the rates of Euler angles is given as

$$\bar{\omega}(t)=\boldsymbol{W}\dot{\Psi} \tag{2}$$

in which the matrix $\boldsymbol{W}\in\Re^{3\times3}$ is defined as

$$\boldsymbol{W}(\Psi)=\begin{bmatrix}0 & -S_\alpha & C_\beta C_\alpha\\ 0 & C_\alpha & C_\beta S_\alpha\\ 1 & 0 & -S_\beta\end{bmatrix} \tag{3}$$

where the kinematic degeneracy (or singularity) is likely to occur at $\det(\boldsymbol{W})=0$ in which $\boldsymbol{W}$ is rank deficient. In this study, $\boldsymbol{W}$ is assumed a nonsingular matrix over any Ψ of interest so that a singular point is eliminated. Then it is known that $\dot{\boldsymbol{R}}(t)=\nabla(\bar{\omega})\boldsymbol{R}$ or $\nabla(\bar{\omega})=\dot{\boldsymbol{R}}\boldsymbol{R}^T$, where the skew-symmetric matrix function ∇ is given by

$$\nabla=[\bar{\omega}\times]=\begin{bmatrix}0 & -\omega_z & \omega_y\\ \omega_z & 0 & -\omega_x\\ -\omega_y & \omega_x & 0\end{bmatrix}$$

where $\bar{\omega}=col(\omega_x,\ \omega_y,\ \omega_z)$; thus $\nabla=\{\nabla\in\Re^{3\times3}: \nabla^T=-\nabla\}$, with $\nabla\in so(3)\subset\Re^{3\times3}$. With the definition given above, the generalized velocity vectors (or twists) are related as

$$\boldsymbol{v}=\boldsymbol{S}(\boldsymbol{p})\dot{\boldsymbol{p}},\ \text{with}\ \boldsymbol{p}=(\boldsymbol{r},\ \boldsymbol{R})\in SE(3),\quad n=6 \tag{4}$$

where $\boldsymbol{v}=[\dot{\boldsymbol{r}}^T\bar{\omega}^T]^T\in\Re^6$ and $\dot{\boldsymbol{p}}=[\dot{\boldsymbol{r}}^T\dot{\Psi}^T]^T\in\Re^6$, with $\boldsymbol{S}=\begin{bmatrix}\boldsymbol{E}_{3\times3} & \boldsymbol{0}_{3\times3}\\ \boldsymbol{0}_{3\times3} & \boldsymbol{W}\end{bmatrix}\in\Re^{6\times6}$. Clearly, $\boldsymbol{S}$ is also a nonsingular matrix. Then the forward kinematics is given in a unique manner as

$$\boldsymbol{p}=\boldsymbol{h}(\boldsymbol{q}) \tag{5}$$

where $\boldsymbol{h}(\circ):\Re^n\rightarrow\Re^n$(or $SE(3)$, $n=6$) represents a mapping from the joint space to the end-effector space. And this transformation is continuous, invertible, and twice differentiable with respect to $\boldsymbol{q}$ (that is, a C^2 or often smooth function C^∞). By chain rule of differentiation, the velocity relation is then given by

$$\boldsymbol{v}=\boldsymbol{J}_v\dot{\boldsymbol{q}} \tag{6}$$

with $\boldsymbol{J}_v=\boldsymbol{SJ}$ and $\boldsymbol{J}=\partial\boldsymbol{h}/\partial\boldsymbol{q}$, where $\boldsymbol{J}_v\in\Re^{n\times n}$ is the standard Jacobian with a full rank, and a unique inverse mapping exists if $\boldsymbol{J}_v$ is a nonsingular matrix with a maximal rank.

If a set of $m(<n)$ time-varying hypersurfaces are imposed on the end-effector by external constraint surfaces, then its algebraic equation can be expressed as

$$\boldsymbol{\Phi}_f(\boldsymbol{p})=[\phi_{f1}(\boldsymbol{p})\cdots\phi_{fm}(\boldsymbol{p})]^T=\boldsymbol{0}_m\ \text{or}\quad \phi_{fi}(\boldsymbol{p})=0,\ (i=1,\ \cdots,\ m) \tag{7}$$

where $\boldsymbol{\Phi}_f\in C^2\{SE(3)\rightarrow\Re^m\}$ with $\phi_{fi}\in C^2\{SE(3)\rightarrow\Re^1\}$ are the natural constraints resulting from the geometric characteristics of task configurations. The gripper motion restrictions described in (7) are called holonomic (differentiable-integrable) constraints in the literature. (khatib, 1987;Yoshikawa, 1987;McClamroch, et. al., 1988) On the other hand, in order to specify the desired motion of the end-effector, a set of $(n-m)$ mutually independent artificial constraints are also introduced as(Mason, 1981;Raibert, et. al., 1981;Khatib, 1987;Yoshikawa, 1987)

$$\boldsymbol{\Phi}_p(\boldsymbol{p})=[\varphi_{p1}(\boldsymbol{p})\cdots\varphi_{p(n-m)}(\boldsymbol{p})]^T \tag{8}$$

with $\boldsymbol{\Phi}_p\in C^2\{SE(3)\rightarrow\Re^{(n-m)}\}$, such that the constraint surface variables ($\boldsymbol{\Phi}_f$ and $\boldsymbol{\Phi}_p$), which are subsets in space $\Re^n$, are mutually independent and at least twice differentiable functions with respect to $\boldsymbol{q}$.

Using the Euler-Lagrange's formulation, the constrained robot dynamics can be expressed as

$$\boldsymbol{M}(\boldsymbol{q};\Theta)\ddot{\boldsymbol{q}}+\boldsymbol{C}(\boldsymbol{q},\ \dot{\boldsymbol{q}};\Theta)\dot{\boldsymbol{q}}+\boldsymbol{G}(\boldsymbol{q};\Theta)+\boldsymbol{T}_c=\boldsymbol{T},\ \forall t\geq0\ \text{with}\ \boldsymbol{\Phi}_f(\boldsymbol{p})=\boldsymbol{0}_m\ \text{and}\ \boldsymbol{p}=\boldsymbol{h}(\boldsymbol{q}) \tag{9}$$

where $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$, and $\ddot{\boldsymbol{q}}\in\Re^n$ are the vectors representing the joint positions, velocities, and accelerations, respectively; $\boldsymbol{M}(\boldsymbol{q};\Theta)\in\Re^{n\times n}$ is an inertia matrix; $\boldsymbol{C}(\boldsymbol{q},\ \dot{\boldsymbol{q}};\Theta)\in\Re^{n\times n}$ is a matrix function denoting the centripetal and Coriolis effects; $G(q;\Theta)\in\Re^n$ is the vector containing gravitational torques; $T\in\Re^n$ is the control input vector; $T_c\in\Re^n$ represents the vector of the contact forces between the end-effector and the external objects, and its functional structure will be specified later. And all robot parameters, such as link lengths, link masses, and moments of inertia, are lumped together into a parameter vector $\Theta\in\Re^s$.

It is well known that the system dynamics (9) with all revolute-type joints satisfies the following fundamental properties.(Ortega, et. al., 1989; Reed, et. al., 1989; Sadegh, et. al., 1990)

[P1]: M is a symmetric and positive-definite matrix, i.e., $M=M^T>0$. Furthermore, M and M^{-1} are uniformly bounded above and below, for example,

$$\lambda_{\min}(M)E\le M\le\lambda_{\max}(M)E,\ \forall(q,\ \Theta)$$
$$\text{with } \lambda_{\max}(M)<\infty$$

[P2]: $(M-2C)$ is a skew-symmetric matrix with a suitable definition of C, and the matrix C is upper-bounded by $\|C\|\le\alpha\|\dot{q}\|$, where $\alpha(>0)$ is a scalar constant.

[P3]: A part of the dynamic structure (9) is linear in terms of a suitably defined set of the dynamic parameters,

$$M(q;\Theta)z+C(q,\ \dot{q};\Theta)x+G(q;\Theta)$$
$$=Y(q,\ \dot{q},\ x,\ z)\Theta$$

where $Y\in\Re^{n\times s}$ is a regressor matrix which depends on the known functions of $(q,\ \dot{q},\ x,\ z)\in\Re^n$; Θ is the vector of unknown system parameters of interest.

2.2 A new system dynamics and problem formulation

This subsection addresses an efficient approach to the problems of converting the joint-space model to the task-space one. First, the subvectors Φ_f and Φ_t are combined to generate a complete set of the generalized position vector in Π_c as

$$X_c=[x_f^Tx_p^T]^T,\ X_c\in\Re^n \tag{10}$$

where $x_f=\Phi_f(p)$ and $x_p=\Phi_p(p)$, with $x_f\in\Re^m$ and $x_p\in\Re^{(n-m)}$. In this formulation, the natural constraints are orthogonal to the artificial constraints. Thus the n-dimensional task space can be split into two mutually orthogonal subspaces; the m-dimensional contact force subspace and the $(n-m)$-dimensional position subspace. Then the corresponding velocity vector can be obtained by

$$\dot{X}_c=J_\Phi v \tag{11}$$

with $J_\Phi=[J_f^TJ_p^T]^T\in\Re^{n\times n}$, where some vectors and matrices are given by

$$J_f=\partial\Phi_f/\partial p\in\Re^{m\times n},\ \text{with}$$
$$[\partial\phi_{fi}/\partial p]^T\in\Re^{n\times 1}\ (i=1,\ \cdots,\ m)$$
$$J_p=\partial\Phi_p/\partial p\in\Re^{(n-m)\times n},\ \text{with}$$
$$[\partial\varphi_{pj}/\partial p]^T\in\Re^{n\times 1}\ (i=1,\ \cdots,\ n-m)$$

in which the matrix J_Φ is the constraint Jocobian transformation. As the constraint equations are mutually independent, the non-square submatrices J_f and J_p have maximal ranks (or full row ranks) over any p, i.e., $rk(J_f)=m$ and $rk(J_p)=(n-m)$, respectively. And a vector $[\partial\phi_{fi}/\partial p]^T$ specifies the orthogonal direction to the local surface at p, that is $[\partial\phi_{fi}/\partial p]\bullet v=0$, with $\dot{x}_p=J_p\dot{v}$. Clearly, the motion vector v lies in the null space of the vector space spanned by $\{[\partial\phi_{f1}/\partial p]^T,\ \cdots,\ [\partial\phi_{fm}/\partial p]^T\}$. It is especially important to note that the row vectors of J_f and J_p span the normal subspace and its orthogonal complement subspace in $\Re^n$, respectively. As a result, the following orthogonality condition holds:

$$J_p\bullet J_f^T$$
$$=0_{(n-m)\times m}\ \text{or}\ J_f\bullet J_p^T=0_{m\times(n-m)} \tag{12}$$

This is equivalent to

$$[\partial\varphi_{pj}/\partial p]\bullet[\partial\phi_{fi}/\partial p]^T=0\ \text{or}$$
$$[\partial\phi_{fi}/\partial p]\bullet[\partial\varphi_{pj}/\partial p]^T=0$$

which also implies that $rs(J_f^T)\subseteq ns(J_p)$ or $rs(J_p^T)\subseteq ns(J_f)$. Since $rs(J_f^T)$ and $rs(J_p^T)$ are the orthogonal complement of each other in the n-dimensional vector space (U), the constraint space can be decomposed into a direct sum ($\oplus$) of two subspaces as

$$\Re^n=rs(J_p^T)\oplus rs(J_f^T),\ \text{with}$$
$$rs(J_p^T)\cap rs(J_f^T)=\{0\} \tag{13}$$

Thus the dimension of the vector space (U) is given by

$$n=dim\{rs(J_p^T)\oplus rs(J_f^T)\}$$
$$=dim\{rs(J_p^T)\}+dim\{rs(J_f^T)\}$$

As a consequence, the constraint frame has the following set of vectors as its basis

$$\{[\partial\phi_{fi}/\partial \boldsymbol{p}]^T (i=1, \cdots, m); \quad [\partial\varphi_{pj}/\partial \boldsymbol{p}]^T (j=1, \cdots, n-m)\} \tag{14}$$

By virtue of the above results, the $rs(\boldsymbol{J}_p{}^T)$ specifies the motion-controlled subspace, while the $rs(\boldsymbol{J}_f{}^T)$ represents the contact force subspace. Furthermore, we can also define a set of new vectors by normalizing the row spaces of $\boldsymbol{J}_f$ and $\boldsymbol{J}_p$, namely

$$\{[\partial\phi_{fi}/\partial \boldsymbol{p}]/\|\partial\phi_{fi}/\partial \boldsymbol{p}\|(i=1, \cdots, m); \quad [\partial\varphi_{pj}/\partial \boldsymbol{p}]/\|\partial\varphi_{pj}/\partial \boldsymbol{p}\|(j=1, \cdots, n-m)\}$$

then the n-dimensional constraint space has a set of unit vectors given above as its new basis.

Now, combining Eqs. (5) and (7) yields

$$\Omega(\boldsymbol{q}) = \boldsymbol{0}_m \tag{15}$$

where $\Omega = \Phi_f(\boldsymbol{h}(\boldsymbol{q})) : C^2(\Re^n \rightarrow \Re^m)$. Then the velocity constraint equation can be obtained as

$$\boldsymbol{J}_f \boldsymbol{v} = \boldsymbol{J}_\Omega \dot{\boldsymbol{q}} = \boldsymbol{0}_m \tag{16}$$

where $\boldsymbol{J}_\Omega (= \boldsymbol{J}_f \boldsymbol{J}_v \in \Re^{m \times n})$ is the constraint Jocobian with a full row rank. As a result, the geometric constraints imposed on the robot end-effector can be considered as restricting its joint-space motion to the following constraint manifold only;

$$\boldsymbol{Q}_M = \{(\boldsymbol{q}, \dot{\boldsymbol{q}}) \in \Re^n : \Omega(\boldsymbol{q}) = \boldsymbol{0}_m, \boldsymbol{J}_\Omega \dot{\boldsymbol{q}} = \boldsymbol{0}_m\}$$

As we shall see later, this condition leads to the dimension reduction of the system, that is, the overall system has only $(n-m)$ DOF of mobility.

In case of a non-rigid (or soft) contact, the constraint forces $(\boldsymbol{f} \in \Re^m)$ are given by

$$\boldsymbol{f} = \begin{cases} \boldsymbol{0}_m & \text{if } \boldsymbol{x}_{fa} < \boldsymbol{x}_f \\ \boldsymbol{k}_s(\boldsymbol{x}_{fa} - \boldsymbol{x}_f) & \text{if } \boldsymbol{x}_{fa} \geq \boldsymbol{x}_f \end{cases} \tag{17}$$

where $\boldsymbol{x}_{fa} \in \Re^m$ indicates the actual end-effector position, which is aligned with the normal direction; $\boldsymbol{x}_f$ is the undeformed reference position of the environment; and the matrix $\boldsymbol{k}_s$ represents the $(m \times m)$ equivalent stiffness with $rk(\boldsymbol{k}_s) = m$. And assuming that the external environment is homogeneous, $\boldsymbol{k}_s$ can be chosen as a positive-definite matrix, i.e., $\boldsymbol{k}_s = diag[k_{s1}, \cdots, k_{sm}]$, where $k_{si}(>0)$ represents the stiffness along dimension i, and a base of $rs(\boldsymbol{k}_s)$ denotes the orthogonal vectors to the contact surface. In fact, the contact forces can be computed by either a wrist-mounted force/torque sensor or the position error measurement in the contact surface. The generalized contact forces (or wrenches) corresponding to external constraints with the assumption of frictionless contact can be expressed in the operational space as

$$\bar{\boldsymbol{F}}_p = \boldsymbol{J}_f{}^T \boldsymbol{f} \tag{18}$$

where the elastic passive environment can generate the reaction forces along the normal directions only. In addition, a wrench and a twist are reciprocal, i.e., $\bar{\boldsymbol{F}}_p \bullet \boldsymbol{v}^T = 0$. By the principle of virtual work, the contact forces can be written in the joint-space as

$$\boldsymbol{T}_c = \boldsymbol{J}_v{}^T \bar{\boldsymbol{F}}_p = \boldsymbol{J}_\Omega{}^T \boldsymbol{f}$$

Therefore, the joint space dynamics (9) can be rewritten in a set of mixed differential and algebraic equations (DAEs)

$$\begin{aligned} &\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{C}\dot{\boldsymbol{q}} + \boldsymbol{G} + \boldsymbol{J}_\Omega{}^T \boldsymbol{f} = \boldsymbol{T} \\ &\Omega(\boldsymbol{q}) = \boldsymbol{0}_m \end{aligned} \tag{19}$$

From Eqs. (6) and (11), we have

$$\dot{\boldsymbol{X}}_c = \boldsymbol{J}_c \dot{\boldsymbol{q}} \tag{20}$$

$$\ddot{\boldsymbol{X}}_c = \dot{\boldsymbol{J}}_c \dot{\boldsymbol{q}} + \boldsymbol{J}_c \ddot{\boldsymbol{q}} \tag{21}$$

where $\boldsymbol{J}_c (= \boldsymbol{J}_\Omega \boldsymbol{J}_v \in \Re^{n \times n})$ has a full rank. The corresponding joint-space variables are given as

$$\dot{\boldsymbol{q}} = \boldsymbol{J}_c{}^{-1} \dot{\boldsymbol{X}}_c \tag{22}$$

$$\ddot{\boldsymbol{q}} = \boldsymbol{J}_c{}^{-1} \ddot{\boldsymbol{X}}_c + \dot{\boldsymbol{J}}_c{}^{-1} \dot{\boldsymbol{X}}_c \tag{23}$$

where the fact that $\dot{\boldsymbol{J}}_c{}^{-1} = -\boldsymbol{J}_c{}^{-1} \dot{\boldsymbol{J}}_c \boldsymbol{J}_c{}^{-1}$ has been utilized (see Appendix A for the proof). After substituting Eqs. (22) and (23) into (19) and permultiplying both sides of the resultant equation by $\boldsymbol{J}_c{}^{-T}$, we obtain

$$\begin{aligned} &\boldsymbol{M}_c(\boldsymbol{X}_c; \Theta)\ddot{\boldsymbol{X}}_c + \boldsymbol{C}_c(\boldsymbol{X}_c, \dot{\boldsymbol{X}}_c; \Theta)\dot{\boldsymbol{X}}_c \\ &+ \boldsymbol{G}_c(\boldsymbol{X}_c; \Theta) + \boldsymbol{J}_c{}^{-T} \boldsymbol{J}_\Omega{}^T \boldsymbol{f} = \boldsymbol{F} \end{aligned} \tag{24}$$

where

$$\begin{aligned} &\boldsymbol{M}_c = \boldsymbol{J}_c{}^{-T} \boldsymbol{M} \boldsymbol{J}_c{}^{-1} \\ &\boldsymbol{C}_c = \boldsymbol{J}_c{}^{-T} \boldsymbol{C} \boldsymbol{J}_c{}^{-1} + \boldsymbol{J}_c{}^{-T} \boldsymbol{M} \dot{\boldsymbol{J}}_c{}^{-1} \\ &\boldsymbol{G}_c = \boldsymbol{J}_c{}^{-T} \boldsymbol{G} \\ &\boldsymbol{F} = \boldsymbol{J}_c{}^{-T} \boldsymbol{T} \end{aligned}$$

For the sake of further analysis, the identity matrix can be partitioned such that

$$\boldsymbol{E}_{n \times n} = [\bar{\boldsymbol{E}}_f : \bar{\boldsymbol{E}}_p]$$

where

$$\bar{E}_f=[E^T{}_{m\times m}\mathbf{0}]^T,\ \bar{E}_f\in\Re^{n\times m}$$
$$\bar{E}_p=[\mathbf{0}E^T{}_{(n-m)\times(n-m)}]^T,\ \bar{E}_p\in\Re^{n\times(n-m)}$$

with $\bar{E}_p{}^T\bar{E}_f=\mathbf{0}$ and $\bar{E}_f{}^T\bar{E}_f=E_{m\times m}$. Using the identity $J_c{}^{-T}J_\Omega{}^T=\bar{E}_f$, the robot dynamics with holonomic constraints can be finally written in the constraint-space as

$$\begin{aligned}&M_c(X_c;\Theta)\ddot{X}_c+C_c(X_c,\dot{X}_c;\Theta)\dot{X}_c\\&+G_c(X_c;\Theta)+\bar{E}_f f=F\\&x_f=\mathbf{0}\end{aligned}\tag{25}$$

where $X_c=\bar{E}_p x_p$. It should be noted that the constraints equations as well as the constraint forces are expressed in simple forms under the new coordinate system. The motion of the entire system is actually governed by the independent variables x_p (i.e., a minimal-order governing equation by the tangent motion x_p). Now, the position- and force-controlled subsystems can be readily decoupled, that is,

$$\begin{aligned}&F_p=\bar{E}_p{}^TM_c\bar{E}_p\ddot{x}_p+\bar{E}_p{}^TC_c\bar{E}_p\dot{x}_p+\bar{E}_p{}^TG_c\\&F_f=\bar{E}_f{}^TM_c\bar{E}_p\ddot{x}_p+\bar{E}_f{}^TC_c\bar{E}_p\dot{x}_p+\bar{E}_f{}^TG_c\\&\quad+f\\&x_f=\mathbf{0}\end{aligned}$$

Consequently, the first subsystem represents the reduced-order equations of motion which contains no contact forces (i.e., purely kinetic differential equations), while the other subsystem is used to regulate the generalized contact forces.

Some essential properties of the transformed dynamics (25) are summarized as follows: (Han, et. al., 1990; Carelli, et. al., 1990; Fossen, et. al., 1991; You, 1994; Sastry, et. al., 1989)

[P1]: M_c is a symmetric positive-definite matrix. Moreover, both M_c and $M_c{}^{-1}$ are uniformly bounded above and below.

[P2]: $\bar{N}_c=(\dot{M}_c-2C_c)$ is a skew-symmetric matrix.

Proof: See Appendix B for the proof.

[P3]: A part of the dynamics (25) is still linear in terms of suitably selected set of the parameter vector ($\Theta\in\Re^r$), namely,

$$\begin{aligned}&M_c(X_c;\Theta)z+C_c(X_c,\dot{X}_c;\Theta)x\\&+G_c(X_c;\Theta)=Y_c(X_c,\dot{X}_c,x,z)\Theta\end{aligned}$$

where x and $z\in\Re^n$; $Y_c\in\Re^{n\times r}$ is a regressor matrix.

In the subsequent section, assuming that the desired state variables (x_{pd}, $\dot{x}_{pd}\ddot{x}_{pd}$, f_d) to be tracked are all continuous and bounded functions, a hybrid (position/force) controller is formulated.

3. Design of a Hybrid Control Algorithm

Before controller design, a number of tracking error vectors are defined as follows. The position (or motion) tracking error vector e_p is given by

$$e_p=x_p-x_{pd}\tag{26}$$

where $x_{pd}\in\Re^{(n-m)}$ is the vector of the desired position trajectories. The constraint force tracking errors are defined as

$$e_f=f-f_d \text{ and } e_F=\int_0^t e_f(\tau)d\tau\tag{27}$$

where $f_d\in\Re^m$ is the desired contact force vector; $e_F\in\Re^m$ is the vector of the accumulated force errors (or momentum signals). The reference position/force tracking error vector, $\dot{X}_{cr}\in\Re^n$, is defined as

$$\dot{X}_{cr}=[\dot{x}_{fr}{}^T\dot{x}_{pr}{}^T]^T\tag{28}$$

with $\dot{x}_{fr}=k_Fe_F$ and $\dot{x}_{pr}=\dot{x}_{pd}-k_pe_p$, where the gain matrices can be selected as positive-definite, that is, $k_p=k_pE(k_p>0)$, and $k_F=k_FE(k_F>0)$. The sliding variable vector, $X_{cs}\in\Re^n$, is defined as

$$X_{cs}=\dot{X}_c-\dot{X}_{cr}=[x_{fs}{}^Tx_{ps}{}^T]^T\tag{29}$$

with $x_{fs}=-k_Fe_F$ and $x_{ps}=\dot{e}_p+k_pe_p$. It is worth noting that $\dot{x}_{fr}$ and x_{fs} are always orthogonal to $\dot{x}_{pr}$ and x_{ps}, respectively. Actually, they are the orthogonal complement vectors each other in $\Re^n$.

Lemma: Let $p(t):\Re^+\rightarrow\Re^n$ be a uniformly continuous functions, then for any $c_0>0$,

$$\lim_{t\to\infty}p(t)=\mathbf{0} \text{ if and only if}$$
$$\lim_{t\to\infty}\int_t^{t+c}p(\tau)d\tau=\mathbf{0} \text{ for all } 0<c\le c_0$$

Proof: See Refs. (Sadegh, et. al., 1990; You, 1994) for more details.

Let $\tilde{\Theta}(t)=\hat{\Theta}(t)-\Theta$ be the parameter error vector, where $\hat{\Theta}(t)$ denotes the current estimates of Θ. In this paper, the circumflex $(\hat{\circ})$ represents the adaptive estimates of $(\circ)$ and the corresponding estimation error $(\tilde{\circ})=(\hat{\circ})-(\circ)$. Utilizing [P3] in (25), we also define the following functions:

$$\begin{aligned} &M_c(X_c;\Theta)\ddot{X}_{cr}+C_c(X_c,\dot{X}_c;\Theta)\dot{X}_{cr} \\ &+G_c(X_c;\Theta) \\ &=\bar{Y}_c(X_c,\dot{X}_c,\dot{X}_{cr},\ddot{X}_{cr})\Theta \end{aligned} \quad (30)$$

$$\begin{aligned} &\hat{M}_c(X_c;\hat{\Theta})\ddot{X}_{cr}+\hat{C}_c(X_c,\dot{X}_c;\hat{\Theta})\dot{X}_{cr} \\ &+\hat{G}_c(X_c;\hat{\Theta}) \\ &=\bar{Y}_c(X_c,\dot{X}_c,\dot{X}_{cr},\ddot{X}_{cr})\hat{\Theta} \end{aligned} \quad (31)$$

where $\bar{Y}_c\in\Re^{n\times r}$ is a regressor matrix which is no longer a function of $\ddot{x}_p$(or $\ddot{X}_c$).

The design objective is to determine a set of control torques so that the actual system responses of interest (x_p, f) simultaneously track the reference (or desired) values (x_{pd}, f_d) as closely as possible. Consider the following hybrid control algorithm for the robot model containing closed-chain mechanism,

$$\begin{aligned} F=&\bar{Y}_c(X_c,\dot{X}_c,\dot{X}_{cr},\ddot{X}_{cr})\hat{\Theta}-KX_{cs} \\ &+\bar{E}_f(f_d+k_fe_F) \end{aligned} \quad (32)$$

where K and k_f are positive-definite gain matrices with appropriate dimensions, for example, $k_f=k_fE(k_f>0)$. Then the adaptation mechanism (Reed, et. al., 1989) is chosen as

$$\dot{\hat{\Theta}}=-U^{-1}(\bar{Y}_c^TX_{cs}+\sigma\hat{\Theta}) \quad (33)$$

where $U(=U^T>0)$ is an adaptation gain matrix, and the term $\sigma(\circ):\Re^r\rightarrow R^+$ is selected as

$$\sigma(\hat{\Theta})=\begin{cases} 0, & \|\hat{\Theta}\|\le\Theta_0 \\ \sigma_0, & \|\hat{\Theta}\|\ge\Theta_0 \end{cases} \quad (34)$$

where the constants $\sigma_0(>0)$ and $\Theta_0(>\|\Theta\|>0)$ are some design parameters. Note that σ is adopted to ensure the robustness of the adaptive algorithm in the presence of uncertainties. Since the control input vector F is viewed as the hypothetical force acting on the constraint space, the generalized contact forces are eventually converted to the joint torques by $T=J_c^TF$. An overall control scheme is shown in Fig. 3.

After substituting the control law (32) with (33) into Eq. (25) and subtracting (30) on the both sides of the resulting equation, we obtain the closed-loop error dynamics as

$$\begin{aligned} M_c\dot{X}_{cs}=&-C_cX_{cs}+\bar{Y}_c\tilde{\Theta}-KX_{cs} \\ &+\bar{E}_f(e_f+k_fe_F) \end{aligned} \quad (35)$$

Now the stability and the tracking properties of the closed-loop system are analyzed in the following.

Theorem: Provided that some or all robot parameters (Θ) in (25) are unknown, then the closed-loop system (35) with the control law (32) is globally stable in the sense that the responses of some state variables will be zero asymptotically, namely

$$x_p\rightarrow x_{pd} \text{ and } f\rightarrow f_d \text{ as } t\rightarrow\infty$$

and the parameter estimation errors $\tilde{\Theta}$ are ultimately bounded.

Proof: Define a Lyapunov function candidate, $V:(t,\bar{X})\in\Re^+\times\Re^{(n-m+r)}\rightarrow\Re^+$, by

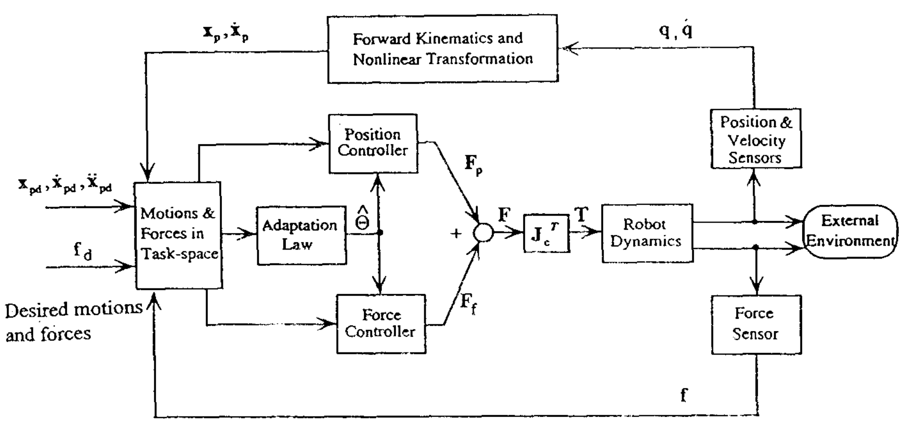

Fig. 3 Block diagram of hybrid control system

$$V=(1/2)\bar{X}^T Q\bar{X} \tag{36}$$

where $\bar{X}=[X_{cs}^T e_F^T \tilde{\Theta}^T]^T$ and $Q=$Block diag $[M_c, k_F, U]$. By using Rayleigh's principle and noting that the matrices M_c, k_F, and U are all positive-definite, we obtain

$$(1/2)\lambda_{\min}(Q)\|\bar{X}\|\leq V\leq(1/2)\lambda_{\max}(Q)\|\bar{X}\|$$

with $\lambda_{\min}(Q)>0$. Thus the scalar function V is a positive-definite. Computing the time derivative of V leads to

$$\begin{aligned}\dot{V}=X_{cs}^T[&-C_cX_{cs}+\bar{Y}_c\tilde{\Theta}-KX_{cs}\\&+\bar{E}_f(e_f+k_fe_F)]+(1/2)X_{cs}^T\dot{M}_cX_{cs}\\&+k_Fe_F^Te_f+\dot{\tilde{\Theta}}^TU\tilde{\Theta}\end{aligned} \tag{37}$$

which can be rewritten as

$$\begin{aligned}\dot{V}=&-X_{cs}^TKX_{cs}+X_{cs}^T\bar{E}_f(e_f+k_fe_F)]\\&+k_Fe_F^Te_f-\sigma\tilde{\Theta}^T\hat{\Theta}\end{aligned}$$

where [P2] in (25) and $\dot{\tilde{\Theta}}=\dot{\hat{\Theta}}$ (assuming $\dot{\Theta}=0$) have been conveniently exploited. By noting the fact that $X_{cs}^T\bar{E}_f=x_{fs}^T$ and $\sigma\tilde{\Theta}^T\hat{\Theta}\geq 0$ (see Appendix C for the proof), it follows that

$$\begin{aligned}\dot{V}&\leq-X_{cs}^TKX_{cs}-k_Fe_F^T(e_f+k_fe_F)\\&\quad+k_Fe_F^Te_f\\&\leq-X_{cs}^TKX_{cs}-k_Fk_fe_F^Te_F\\&\leq-\lambda_{\min}(K)\|X_{cs}\|^2-k_Fk_f\|e_F\|^2\leq 0\end{aligned} \tag{38}$$

Thus, $V\leq V_0=V_{t=0}(\circ)<\infty$, $\forall t\geq 0$, is lower bounded by zero. From (36) and (38), it can be seen that $V\in L_\infty$, and accordingly $X_{cs}\in L_\infty$, $e_F\in L_\infty$, and $\tilde{\Theta}\in L_\infty$. In addition, from (38),

$$\begin{aligned}&\lambda_{\min}(K)\int_0^\infty\|X_{cs}\|^2dt+k_fk_F\int_0^\infty\|e_F\|^2dt\\&\leq V_0-\lim_{t\to\infty}V\leq V_0\end{aligned}$$

which implies that $X_{cs}\in L_2$ and $e_F\in L_2$. On the other hand, due to the fact (Han, et. al., 1990) that $\|e_f\|\leq\|e_p\|$, the constraint force tracking errors (e_f) also remain bounded. In fact, the external environment can be regarded as a passive mechanism, thus it only provides a finite amount of energy. Based on this observation, f is reasonably assumed to be bounded as function of time. Consequently, the matrix $\bar{Y}_c$ can be readily shown to be bounded too. And from (35) and the above results, one obtain $\dot{X}_{cs}\in L_\infty$ and $X_{cs}\in L_\infty\cap L_2$. Using Barbalat's Lemma, (You, 1994;Sastry, et. al., 1989) we get $\lim_{t\to\infty}X_{cs}\to 0$. Evidently, this result shows that $\lim_{t\to\infty}e_F\to 0$ and $\lim_{t\to\infty}(\dot{e}_p+k_pe_p)\to 0$, which in turn implies that

$$e_p(t)=e_p(t_0)\exp[-k_p(t-t_0)] \text{ with}$$
$$\|e_p(t_0)\|<\infty, \text{ and } \int e_f(\tau)d\tau\to 0$$

Hence, it can be concluded that $e_p\to 0$ and $e_f\to 0$ (by Lemma) as $t\to\infty$. Furthermore, the uniform continuity of the signals e_p also leads to $\lim_{t\to\infty}\dot{e}_p=0$.(You, 1994)

4. Conclusions

This paper has presented a unified formulation to the dynamic model and the hybrid control for the constrained robotic manipulator over known contact surfaces. The compact mathematical model has been given in terms of the constraint surface variables. The constraint frame is set up as a direct sum of the force-controlled subspace and (purely kinetic) position-controlled subspace in which the position and force DOF are specified on the tangential and normal directions, respectively. Based on the new reduced dynamic model, a hybrid control algorithm is synthesized, that is, the generalized positions and the contact forces are simultaneously regulated in two orthogonal directions. A rigorous stability analysis of the closed-loop systems has been given by a Lyapunov method. Further, it has been shown that the proposed control law guarantees asymptotic stability of the position tracking errors as well as the contact force errors.

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Appendix

Appendix A

Proof of $\dot{J}_c^{-1} = -J_c^{-1}\dot{J}_c J_c^{-1}$:

Noting that $J_c J_c^{-1} = E$, we obtain $\dot{J}_c J_c^{-1} + J_c \dot{J}_c^{-1} = 0$. Thus it is clear that $\dot{J}_c^{-1} = -J_c^{-1}\dot{J}_c J_c^{-1}$.

Appendix B

Proof of [P2] in Eq. (25):

Assuming that $\bar{N}_c$ is a skew-symmetric, we see that $\bar{x}^T \bar{N}_c \bar{x} = 0$, $\bar{x} \in \Re^n$. Hence, set

$$\begin{aligned}\bar{x}^T \bar{N}_c \bar{x} &= \bar{x}^T \left[\tfrac{d}{dt}(J_c^{-T} M J_c^{-1}) \right. \\ &\quad \left. -2(J_c^{-T} C J_c^{-1} - J_c^{-T} M J_c^{-1} \dot{J}_c J_c^{-1}) \right] \bar{x} \\ &= \bar{x}^T J_c^{-T} (\dot{M} - 2C) J_c^{-1} \bar{x}\end{aligned}$$

Since the matrix $(\dot{M} - 2C)$ is a skew-symmetric, we have $\bar{Z}^T (\dot{M} - 2C) \bar{Z} = 0$, with $\bar{Z} = J_c^{-1}\bar{x}$. Therefore, we conclude that the matrix $\bar{N}_c$ is also a skew-symmetric.

Appendix C

Proof of $\sigma\tilde{\Theta}^T\hat{\Theta} \geq 0$: To begin, we note that

$$\begin{aligned}\sigma\tilde{\Theta}^T\hat{\Theta} &= \sigma(\hat{\Theta}^T - \Theta^T)\hat{\Theta} = \sigma(\|\hat{\Theta}\|^2 - \Theta^T\hat{\Theta}) \\ &\geq \sigma\|\hat{\Theta}\|(\|\hat{\Theta}\| - \Theta_0 + \Theta_0 - \|\Theta\|)\end{aligned}$$

Note also that $\sigma\|\hat{\Theta}\|(\|\hat{\Theta}\| - \Theta_0) \geq 0$ and $\Theta_0 \geq \|\Theta\|$, it follows that $\sigma\tilde{\Theta}^T\hat{\Theta} \geq 0$.